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# ON RADICALS OF TRIANGULAR OPERATOR ALGEBRAS

BY

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ABSTRACT

Suppose $\mathcal{B}$ is a type I $C^*$-algebra admitting a diagonal $\mathcal{D}$ in the sense of Kumjian, and let $E$ be the conditional expectation from $\mathcal{B}$ onto $\mathcal{D}$. A subalgebra $\mathcal{A}$ of $\mathcal{B}$ is called **triangular** with diagonal $\mathcal{D}$ if $\mathcal{A} \cap \mathcal{A}* = \mathcal{D}$.

THEOREM: *Under the above assumptions the Jacobson radical of $\mathcal{A}$ equals the intersection of $\mathcal{A}$ with the kernel of the conditional expectation $E$.*

Although the statement of the theorem is coordinate free, the proof requires the use of coordinates in essential ways.

A theorem by Kumjian allows us to represent every $C^*$-algebra admitting a diagonal as the $C^*$-algebra of a certain groupoid. This enables us to apply the techniques of topological groupoids as developed by Renault and Muhly. A very convenient way of expressing a triangular subalgebra of the $C^*$-algebra of a $\mathbf{T}$-groupoid is given by the Spectral Theorem for Bimodules, due to Qui, which is a descendent of the Spectral Theorem for Bimodules due to Muhly and Solel, and to Muhly, Saito and Solel in the context of von Neumann algebras.

## 1. Introduction

Kumjian introduced the notion of "diagonal subalgebra" of a $C^*$-algebra in [K]. This generalizes the concept of the diagonal in the case when the $C^*$-algebra is $M_n(\mathrm{C})$, i.e. the algebra of $n \times n$ complex matrices. If $\mathcal{D}$ is an abelian $C^*$-subalgebra of a unital $C^*$-algebra $\mathcal{A}$, then $\mathcal{D}$ is said to be diagonal in $\mathcal{A}$ if $1 \in \mathcal{D}$ and if: (i) there is a faithful conditional expectation $E$: $\mathcal{A} \to \mathcal{D}$; and (ii) $\mathrm{Ker}E = (\mathrm{span}N_f(\mathcal{D}))^-$, where $N_f(\mathcal{D})$ denotes the set of all free normalizers of $\mathcal{D}$, i.e. the set of elements $a$ in $\mathcal{A}$ such that $a\mathcal{D}a*$ and $a^*\mathcal{D}a$ are in $\mathcal{D}$, and $a^2 = 0$. If $\mathcal{A}$ is

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not unital, and if $\tilde{\mathcal{A}}$ denotes the $C^*$-algebra obtained from $\mathcal{A}$ by adjoining a unit, then $\mathcal{D}$ is said to be diagonal in $\mathcal{A}$ if $\tilde{\mathcal{D}}$ is diagonal in $\tilde{\mathcal{A}}$.

If $\mathcal{A}$ is a $C^*$-algebra with diagonal $\mathcal{D}$, and if $\mathcal{B}$ is a closed subalgebra of $\mathcal{A}$, then $\mathcal{B}$ is called **triangular** with diagonal $\mathcal{D}$ if $\mathcal{B} \cap \mathcal{B}^* = \mathcal{D}$.

Recall that the **Jacobson radical** of a Banach algebra $\mathcal{B}$ is defined to be the ideal of all elements $b$ in $\mathcal{B}$ such that $ab$ is quasi-nilpotent for all $a$ in $\mathcal{B}$. In this work we study and describe the radical of a triangular subalgebra $\mathcal{B}$ of a $C^*$-algebra $\mathcal{A}$ in the case when $\mathcal{A}$ is a type I $C^*$-algebra. We show that in this case the radical may be described just as in the finite dimensional case. More precisely, our main theorem (Theorem 6.2) states that if $\mathcal{A}$ is a type I $C^*$-algebra admitting a diagonal $\mathcal{D}$ and if $\mathcal{B}$ is a triangular subalgebra of $\mathcal{A}$ with the same diagonal $\mathcal{D}$, then the Jacobson radical of $\mathcal{B}$ equals the intersection of $\mathcal{B}$ with the kernel of the conditonal expectation $E$. That is, the radical of $\mathcal{B}$ coincides with the collection of elements of $\mathcal{B}$ having "zero diagonal". Observe, too, that in this case $\mathcal{B}$ has the Wedderburn property, i.e. the radical of $\mathcal{B}$ is complemented as a Banach space.

Following a result of Kumjian in the above-mentioned work, every $C^*$-algebra admitting a diagonal can be represented as the $C^*$-algebra of a certain groupoid. This result allows us to apply the techniques of topological groupoids as developed by Renault and Muhly. Notice that, although we use groupoids as coordinates, the result does not depend on such coordinates, i.e., the statement is coordinate free. Yet the proof requires coordinates in essential ways.

Next we recall some terminology and notation (with regard to groupoids we follow [M2]) used throughout this paper. Suppose $\mathcal{G}$ is a second countable, locally compact, $r$-discrete principal groupoid. Recall that such a groupoid may be viewed as an equivalence relation over $\overset{\circ}{\mathcal{G}}$, the unit space of $\mathcal{G}$.

Let $\mathcal{E}$ be a locally compact groupoid such that $\mathbf{T}$ acts freely and continuously on $\mathcal{E}$ so that $\mathbf{T}\backslash\mathcal{E}$ is Hausdorff. Suppose that $r(t.x) = r(x)$ and $s(t.x) = s(x)$, so that $(x, y) \in \mathcal{E}^2$ implies $(t.x, t.y) \in \mathcal{E}^2$, and suppose we also have $(t.x)(s.y) = (ts).(xy)$. Then $\mathbf{T}\backslash\mathcal{E}$ has the structure of a groupoid. If $\mathbf{T}\backslash\mathcal{E} = \mathcal{G}$ is principal and $r$-discrete, we call $\mathcal{E}$ an **$r$-discrete principal T-groupoid over $\mathbf{T}\backslash\mathcal{E} = \mathcal{G}$**. Suppose $\mathcal{E}$ is an $r$-discrete principal $\mathbf{T}$-groupoid over $\mathcal{G}$, and let $X = \overset{\circ}{\mathcal{G}}$. Let

$$C_c(\mathcal{G}, \mathcal{E}) = \{f \in C_c(\mathcal{E}), f(t.x) = tf(x)\}$$

and

$$\Omega(\mathcal{G}) = \{U \subseteq \mathcal{G} \mid U - \text{open}, r|U \text{ and } s|U \text{ are } 1-1\}.$$

Write

$$N(\mathcal{E}) := \{f \in C_c(\mathcal{G}, \mathcal{E}) \mid \text{supp}(f) \subseteq U, U \in \Omega(\mathcal{G})\},$$

$$D_c(\mathcal{E}) := \{f \in C_c(\mathcal{G}, \mathcal{E}) \mid \text{supp}(f) \subseteq X\}$$

and

$$N_f(\mathcal{E}) := \{f \in N(\mathcal{E}) \mid f^2 = 0\}.$$

Then $D_c(\mathcal{E})$ is isomorphic to $C_c(X)$, $N(\mathcal{E})$ are the normalizers of $D_c(\mathcal{E})$ and $N_f(\mathcal{E})$ are the free normalizers of $D_c(\mathcal{E})$. We write $D(\mathcal{E})$ for $C_0(X)$. Theorem 3.1 in [K] states that if $\mathcal{E}$ is an $r$-discrete, principal $\mathbf{T}$-groupoid over $\mathcal{G}$, then $D(\mathcal{E})$ is a diagonal subalgebra in $C^*_{\text{red}}(\mathcal{G};\mathcal{E})$. Conversely, if $A$ is a $C^*$-algebra containing a diagonal $D$, then there is an (essentially) unique $r$-discrete, principal $\mathbf{T}$-groupoid $\mathcal{E}$ over $\mathcal{G}$ and an isomorphism $\Phi$ from $A$ onto $C^*_{\text{red}}(\mathcal{G};\mathcal{E})$ such that $\Phi(D) = D(\mathcal{E})$, where $D(\mathcal{E}) \cong C_0(X)$ and $X$ is the diagonal of $\mathcal{G}$.

*Standing convention:* From now on, every time we have a $C^*$-algebra $A$ admitting a diagonal $D$ it will be implied that $A$ is isomorphic to $C^*_{\text{red}}(\mathcal{G};\mathcal{E})$ and $D$ is isomorphic to $D(\mathcal{E})$ without further clarification.

An open subset $\mathcal{P}$ of $\mathcal{G}$ containing $\mathring{\mathcal{G}}$ is called a **partial order** if $\mathcal{P} \circ \mathcal{P} \subseteq \mathcal{P}$ and if $\mathcal{P} \cap \mathcal{P}^{-1} = \mathring{\mathcal{G}}$. If, in addition, $\mathcal{P} \cup \mathcal{P}^{-1} = \mathcal{G}$, then $\mathcal{P}$ is called a **total order**. In [MS] it was shown that if $\mathcal{G}$ is a measurewise amenable groupoid, then every triangular subalgebra $\mathcal{A}$ of $C^*(\mathcal{G})$ is the closure, in the $C^*$-norm, of the set of all functions in $C_c(\mathcal{G})$ supported in $\mathcal{P}$, i.e. if $\mathcal{A}$ is a triangular subalgebra of $C^*(\mathcal{G})$ then $\mathcal{A} = \mathcal{A}(\mathcal{P})$ for some open partial order $\mathcal{P}$ of $\mathcal{G}$. This result, known as the Spectral Theorem for Bimodules, was later generalized by Qiu [Q] for the case of a triangular subalgebra of the $C^*$-algebra of a $\mathbf{T}$-groupoid $\mathcal{E}$ over $\mathcal{G}$, provided it is type I.

*1.1 Remark:* Because our $C^*$-algebras are assumed to be separable, our groupoids are second countable. Therefore every quasi-invariant measure $\mu$ is the direct integral of ergodic quasi-invariant measures. If $C^*(\mathcal{G};\mathcal{E})$ is type I, then every ergodic measure on $\mathring{\mathcal{G}}$ is concentrated in an orbit [Ra]. Renault [Re1] shows that every transitive measure (i.e. every measure concentrated in an orbit) is amenable. Therefore, if $C^*(\mathcal{G};\mathcal{E})$ is type I, then $\mathcal{G}$ is measurewise amenable.

Since we are interested only in groupoids such that $C^*(\mathcal{G};\mathcal{E})$ is type I, we will only deal with measurewise amenable groupoids. Recall that in this case $C^*(\mathcal{G};\mathcal{E}) = C^*_{\text{red}}(\mathcal{G};\mathcal{E})$. ∎

From now on all groupoids $\mathcal{G}$ will be locally compact 2nd countable measurewise amenable $r$-discrete principal groupoids, and we will usually refer to $\overset{\circ}{\mathcal{G}}$ by $X$.

In sections 2 and 3 we present some useful realizations of $\mathcal{A}(\mathcal{P})$. Sections 4, 5 and 6 are devoted to the proof of our main theorem. We proceed in three steps. In section 4, we describe the radical of a triangular subalgebra of an elementary $C^*$-algebra (Theorem 4.1). We then describe the radical of a triangular subalgebra of a continuous trace $C^*$-algebra (Theorem 5.2). In section 6 we conclude the proof (Theorem 6.1) for the general case of a triangular subalgebra of a type I $C^*$-algebra. In section 7 we present some examples and we apply the results of the preceding sections to some specific analytic crossed products.

## 2. $\mathcal{A}(\mathcal{P})$ as the algebra of upper triangular matrices

In this section $\mathcal{G}$ will stand for the trivial groupoid, i.e. $\mathcal{G} = X \times X$ with $X$ discrete and $\lambda$ the counting measure on $X$. For every $u$ in $X$, $[u] = X$. Let $\pi$ be the representation of $C_c(\mathcal{G})$ in $\ell^2([u])$ given by

$$\pi(f)\xi(x) = \sum_{y\in X} f(x,y)\xi(y),$$

for $f$ in $C_c(\mathcal{G})$ and $\xi$ in $\ell^2([u])$. Since $\pi(f)$ is a compact operator, for all $f$ in $C_c(\mathcal{G})$ and $\pi$ is irreducible ([MW1], Lemma 2.4), then $C^*(\mathcal{G},\lambda) \cong \mathcal{K}(\ell^2(X,\lambda))$, where $\mathcal{K}(\ell^2(X,\lambda))$ denotes the ideal of compact operators in $\ell^2(X,\lambda)$. On the other hand, since $\mathcal{G}$ is measurewise amenable, $C^*(\mathcal{G},\lambda) = C^*_{\text{red}}(\mathcal{G})$ ([Rel]), and the operations in the $C^*$-algebra $C^*(\mathcal{G},\lambda)$ may be expressed in the same way as in the $*$-algebra $C_c(\mathcal{G})$ ([Rel], Proposition 4.2).

Let $\mathcal{P}$ be a partial order in $\mathcal{G}$. Observe that $\mathcal{P}$ is, in fact, the graph of a partial order in $X$. Many authors use $(X,\leq)$ for denoting a partially ordered set. Sometimes this notation proves to be more convenient, as in the following case. Given $(X,\leq)$, a subset $E$ of $X$ is called **increasing** if, for each $x \in E$ and $y \in X$, $x \leq y$ implies $y \in E$. Let

$$L(X,\leq) = \{E \mid E \text{ is an increasing subset of } X\},$$

and

$$\mathrm{L}(X,\leq) = \{Q_E |\ E \in L(X,\leq)\},$$

where $Q_E$ denotes multiplication by $1_E$ on $\ell^2(X)$. By [A], (Proposition 1.2.1), $\mathrm{L}(X,\leq)$ is a subspace lattice. Theorem 2.1 below is reminiscent of results in [A], but because the space $X$ involved is discrete, the proofs are easier.

2.1 THEOREM: *Let $\mathcal{A} = \mathcal{A}(\mathcal{P})$ be a triangular subalgebra of $C^*(\mathcal{G})$, where $\mathcal{G} = X \times X$, with $X$ discrete, $\lambda$ the counting measure on $X$, and $\mathcal{P}$ a partial order in $\mathcal{G}$. Let $\pi$ be the representation of $C^*(\mathcal{G})$ on $\ell^2(X)$ defined above. Let $L(X,\leq)$ be the lattice of projections in $l^2(X)$,*

$$L(X,\leq) = \{Q_E |\ E \text{ is an increasing subset of } X\}.$$

*Then*

$$\pi(\mathcal{A}) = \mathcal{K}(l^2(X)) \cap \mathrm{Alg}(L(X,\leq)).$$

*Proof:* We first show that $\pi(\mathcal{A}) \subseteq \mathrm{Alg}(\mathrm{L}(X,\leq))$, i.e. we show that if $E$ is any increasing subset of $X$, and $\xi$ is any function in $l^2(E)$, then

$$\pi(a)\xi \in l^2(E), \quad \text{for all } a \text{ in } \mathcal{A}.$$

Equivalently, we show that $< \pi(a)\xi, \eta >= 0$, for every $a \in \mathcal{A}$, $\xi \in l^2(E)$, and $\eta$ in the orthogonal complement of $l^2(E)$ in $l^2(X)$. Let $\{e_x\}_{x\in X}$ be the canonical basis of $l^2(X)$, i.e.

$$e_x(y) = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{otherwise.} \end{cases}$$

Let $E$ be any increasing subset of $X$, and assume $y \in E$. It suffices to show that

$$< \pi(a)e_y, e_z >= 0 \quad \text{for all } z \text{ not in } E.$$

But

$$< \pi(a)e_y, e_z >= a(z,y),$$

and if $z$ is not in $E$, then $z$ cannot be greater than $y$. This means that $(z,y) \notin \mathcal{P}$, and therefore $a(z,y) = 0$ for all $a$ in $\mathcal{A}$.

Hence $\pi(\mathcal{A}) \subseteq \mathrm{Alg}(\mathrm{L}(X,\leq))$, and since $\pi(C^*(\mathcal{G})) = \mathcal{K}(l^2(X))$, we have that

$$\pi(\mathcal{A}) \subseteq \mathrm{Alg}(\mathrm{L}(X,\leq)) \cap \mathcal{K}(l^2(X)).$$

For the reverse inclusion observe that if $K$ is in $\mathcal{K}(l^2(X))$, then there is an $f$ in $C^*(\mathcal{G})$ such that $\pi(f) = K$. Suppose $K$ is in Alg(L). This means that for every increasing subset $E$ of $X$ and every $x$ in $E$,

$$< Ke_x, e_y >= 0, \quad \text{for all } y \notin E.$$

But then

$$< Ke_x, e_y >=< \pi(f)e_x, e_y >= f(y,x) = 0.$$

This shows that the support of $f$ is contained in $\mathcal{P}$, so the reverse inclusion holds. Thus

$$\pi(\mathcal{A}) = \mathrm{Alg}(\mathrm{L}(X, \leq)) \cap \mathcal{K}(l^2(X)). \qquad \blacksquare$$

### 3. Bundle representation of $\mathcal{A}(\mathcal{P})$

The next result, Theorem 3.1, characterizes $\mathcal{A}(\mathcal{P})$ as the $C_0$ cross-sectional Banach algebra of a certain Banach bundle. To present it, we need to recall certain facts about continuous trace $C^*$-algebras and $C^*$-bundles.

In dealing with Banach bundles we are going to adopt the point of view of [FD]. Recall, however, that when the base space $X$ is either locally compact or precompact, by a Theorem of Dal Soglio–Hérault, the concept of Banach bundles coincides with the concept of continuous fields of Banach spaces given in [Go] and in Chapter 10 of [Di].

We pause here to describe useful realizations of the irreducible representations of $C^*(\mathcal{G}, \mathcal{E})$. The details left unverified here may be found in [MW2]. For each $u$ in $X$, $f$ in $C_c(\mathcal{G}, \mathcal{E})$, and $\xi$ in $l^2([u])$ define:

$$T^u(f)\xi(x) = \sum_{y \in [u]} f(c(x,y))\xi(y)\omega(x,y),$$

where $c$ is a regular Borel cross section to the map $j$: $E \to \mathcal{G}$ and $\omega$ is the associated cocycle on $\mathcal{G}$ with values in $\mathbf{T}$. Then $T^u$ is an irreducible representation of $C^*(\mathcal{G}, \mathcal{E})$. Furthermore if $[u] = [v]$, then $T^u$ is unitarily equivalent to $T^v$ ([MW2], Lemma 3.2).

Recall that a $C^*$-algebra $A$ is continuous trace if the ideal of continuous trace elements of $A$ is dense in $A$. If $A$ is a continuous trace $C^*$-algebra, then its spectrum is locally compact and Hausdorff. Suppose $C^*(\mathcal{G}, \mathcal{E})$ is a continuous trace $C^*$-algebra. Then, by [Di] Theorem 10.5.4, it is isomorphic to the cross sectional $C^*$-algebra $C_0(X, X * \mathcal{B})$, where $X$ is the spectrum of $C^*(\mathcal{G}, \mathcal{E})$ and $\mathcal{B} = \{B(x)\}_{x \in X}$ is a $C^*$-bundle of elementary $C^*$-algebras over $X$. On the other hand, $X$ is isomorphic to $\overset{\circ}{\mathcal{G}}/\mathcal{G}$ ([MW2], Proposition 3.3). Each $B(x)$ is isomorphic to $C^*(\mathcal{G}, \mathcal{E})/\mathrm{Ker}(T^u)$, where $T^u$ is an irreducible representation of $C^*(\mathcal{G}, \mathcal{E})$ such that $[u] = x$.

Let $\mathcal{B}$ be a bundle of $C^*$-algebras over $X$. For each $x$ in $X$, let $B'(x)$ be a closed, not necessarily self-adjoint subalgebra of $B(x)$. Let $\Gamma$ be the set of all continuous cross-sections $f$ of $\mathcal{B}$ such that $f(x)$ is in $B'(x)$ for every $x$ in $X$. Suppose that for every $x$ in $X$ the set $\{f(x) | \ f \in \Gamma\}$ is dense in $B'(x)$. Then with the operations restricted from $\mathcal{B}$, $\mathcal{B}'$ becomes a bundle of Banach algebras over $X$. We call it the **subbundle of Banach algebras over $X$ with fibers** $\{B'(x)\}$.

The following observation will be used in the sequel.

*3.1 Remark:* Suppose $X$ is discrete and $\mathcal{G} = X \times X$. Since every circle bundle over a discrete space is trivial, Proposition 4 of [K] implies that $\mathcal{E} \cong \mathbf{T} \times \mathcal{G}$. Therefore $C^*(\mathcal{G}; \mathcal{E})$ is isomorphic to $C^*(\mathcal{G})$. ∎

3.2 THEOREM: *Let $C^*(\mathcal{G}; \mathcal{E})$ be a continuous trace $C^*$-algebra, let $\mathcal{B}$ be the bundle of elementary $C^*$-algebras over $X$ defined by $C^*(\mathcal{G}; \mathcal{E})$, and let $\Phi$ be the isomorphism taking $C^*(\mathcal{G}; \mathcal{E})$ onto $C_0(X, X * \mathcal{B})$ described above. Let $\mathcal{A}$ be a triangular subalgebra of $C^*(\mathcal{G}; \mathcal{E})$. Then there is a subalgebra bundle $\mathcal{B}'$ over $X$ such that $\Phi$ takes $\mathcal{A}$ onto the $C_0$ cross-sectional Banach algebra $C_0(X, X * \mathcal{B}')$.*

*Proof:* Assume $\mathcal{A}$ is a triangular subalgebra of $C^*(\mathcal{G}; \mathcal{E})$, and let $\mathcal{A} = \mathcal{A}(\mathcal{P})$, where $\mathcal{P}$ is an open partial order in $G$. Let $B'(x)$ be the image of $\mathcal{A}$ in $B(x)$. Then

$$B'(x) = T^u(\mathcal{A}(\mathcal{P})), \quad \text{where } [u] = x.$$

Our claim is that

$$\mathcal{A} = C_0(X, X * \mathcal{B}'), \quad \text{where } \mathcal{B}' = \{B'(x)\}_{x \in X}.$$

We first show that each $B'(x)$ is closed in $B(x)$. Write

$$T^u(f)\xi(x) = \sum_{y \in X} K_u(x, y)\xi(y)$$

where

$$K_u(x, y) = f \circ c(x, y)\omega(x, u, y).$$

If $f$ is an element of $\mathcal{A}$, we have

$$\mathrm{Supp}(K_u) = c^{-1}(\mathrm{Supp}(f)) \cap [u] \times [u],$$

and

$$c^{-1}(\mathrm{Supp}(f)) \subseteq c^{-1}(j^{-1}(\mathcal{P})) = \mathcal{P}.$$

Let

$$\mathcal{P}_{[u]} := \mathcal{P} \cap [u] \times [u].$$

We have

$$\mathrm{Supp}(K_u) \subseteq \mathcal{P}_{[u]}, \quad \text{for all } f \text{ in } \mathcal{A},$$

and therefore

$$T^u(\mathcal{A}(\mathcal{P})) = \mathcal{A}(\mathcal{P}_{[u]}).$$

Observe that $\mathcal{A}(\mathcal{P}_{[u]})$ is actually a subalgebra of $C^*(\mathcal{G}|_{[u]}; \mathcal{E}|_{[u]})$. Since $\mathcal{G}|_{[u]}$ is a transitive and principal subgroupoid of $\mathcal{E}$ with unit space $[u]$, and since $[u]$ is discrete, 3.1 implies that $C^*(\mathcal{G}|_{[u]}; \mathcal{E}|_{[u]})$ is ismorphic to $C^*(\mathcal{G}|_{[u]})$. Since $\mathcal{G}|_{[u]}$ is a transitive subgroupoid of $\mathcal{G}$, Theorem 2.1 shows that

$$T^u(\mathcal{A}(\mathcal{P})) = \mathcal{A}(\mathcal{P}_{[u]}) = \mathcal{K}(l^2([u])) \cap \mathrm{Alg}(\mathrm{L}).$$

Hence, $B'(x)$ is closed in $B(x)$. Observe, too, that if

$$\Gamma_0 = \{\Phi(f) |\, f \in \mathcal{A}(\mathcal{P})\},$$

then $\Gamma_0$ is a collection of continuous cross sections of $X * \mathcal{B}'$ satisfying the conditions of Theorem 13.18 of [FD]. Therefore $\mathcal{B}' = \langle X * \mathcal{B}', \pi\rangle$ is a Banach bundle over $X$. Furthermore $C_0(X, X * \mathcal{B}')$ is a subalgebra of $C_0(X, X * \mathcal{B})$. It is clear that $\mathcal{A}$ may be embedded, via $\Phi$, in $C_0(X, X * \mathcal{B}')$. Suppose that the image of $\mathcal{A}(\mathcal{P})$ by $\Phi$ is a proper subalgebra of $C_0(X, X * \mathcal{B}')$, i.e. that there is a non-zero continuous cross section $\varphi$ such that $\varphi(x) \in B'(x)$ for all $x$, $\varphi(x)$ tends to zero at infinity, and that there is a non-zero element $f$ of $C^*(\mathcal{G}; \mathcal{E})$, such that $f \notin \mathcal{A}(\mathcal{P})$, and such that $\Phi(f) = \varphi$. This means that there is an element $\gamma \notin j^{-1}(\mathcal{P})$ such that $f(\gamma) \neq 0$. Let $u = r(\gamma)$. Then

$$B([T^u]) = C^*(\mathcal{G}; \mathcal{E})/\mathrm{Ker}(T^u)$$

and $B'([T^u])$ is the image of $\mathcal{A}(\mathcal{P})$ in $B([T^u])$. But since $f(\gamma) \neq 0$, $f$ is not in the kernel of $T^u$, and since $T^u(f)$ is not in $T^u(\mathcal{A}(\mathcal{P}))$ this implies that $\Phi(f)(x) \notin B'(x)$, $x = [T^u]$. This contradiction shows that, in fact,

$$\Phi(\mathcal{A}) = C_0(X, X * \mathcal{B}'). \quad \blacksquare$$

## 4. The radical of a triangular subalgebra of an elementary $C^*$-algebra

4.1 THEOREM: *Let $\mathcal{A}$ be a triangular subalgebra of $C^*(\mathcal{G})$, where $\mathcal{G} = X \times X$, with $X$ discrete. Then the Jacobson radical of $\mathcal{A}$ equals the set*

$$\{a \in \mathcal{A} |\ a(x,x) = 0 \text{ for all } x \text{ in } X\}.$$

*Proof:* We know that there is an open partial order $\mathcal{P}$ in $\mathcal{G}$ such that $\mathcal{A} = \mathcal{A}(\mathcal{P})$. We begin by showing that the desired result holds when $\mathcal{P}$ is a total order in $\mathcal{G}$. Let $\pi$ be the representation of $C^*(\mathcal{G})$ into $l^2(X)$ given by

$$\pi(f)\xi(x) = \sum_{y \in X} f(x,y)\xi(y).$$

By (2.1), we know that

$$\pi(\mathcal{A}) = \mathcal{K}(l^2(X)) \cap \mathrm{Alg}(\mathrm{L}),$$

where $\mathrm{L} = \{Q_E | E \text{ is an increasing subset of } X\}$. Since $\mathcal{P}$ is assumed to be a total order, Alg(L) is a nest algebra. Applying [Da], Corollary 6.9, a compact operator in a nest algebra belongs to its radical if and only if it is zero on the atomic part of the diagonal. But since in this case the diagonal is purely atomic, the operator must be zero on the diagonal. Let

$$\mathcal{A}_0(\mathcal{P}) := \{a \in \mathcal{A}(\mathcal{P}) |\ a(x,x) = 0 \text{ for all } x \text{ in } X\}.$$

Since $\pi(\mathcal{A}(\mathcal{P}))$ is an ideal of Alg(L), we have

$$\begin{aligned} J(\pi(\mathcal{A})) &= \pi(\mathcal{A}) \cap J(\mathrm{Alg}(\mathrm{L})) \\ &= \{a \in \mathcal{A} |\ a(x,x) = 0 \text{ for all } x \text{ in } X\} \\ &= \mathcal{A}_0(\mathcal{P}), \end{aligned}$$

where $J$ denotes the Jacobson radical. This shows that the desired result holds when $\mathcal{P}$ is a total order in $\mathcal{G}$.

Let $\mathcal{P}$ denote now a partial order and consider $\mathcal{A}_0(\mathcal{P})$. It is an ideal of $\mathcal{A}(\mathcal{P})$ and hence it is enough to show that every element of $\mathcal{A}_0(\mathcal{P})$ is quasi-nilpotent to prove that it is contained in the Jacobson radical of $\mathcal{A}(\mathcal{P})$ ([BD], Proposition III. 16 iii)). $\mathcal{P}$ can be extended to a total order $\mathcal{P}_T$. Clearly we have

$$\mathcal{A}_0(\mathcal{P}) \subseteq \mathcal{A}_0(\mathcal{P}_T).$$

By what was just proved,

$$A_0(\mathcal{P}_T) = J(\mathcal{A}(\mathcal{P}_T)).$$

Therefore every element in $A_0(\mathcal{P}_T)$ is quasi-nilpotent. In particular, $A_0(\mathcal{P})$ consists solely of quasi-nilpotent elements. Thus

$$\mathcal{A}_0(\mathcal{P}) \subseteq J(\mathcal{A}(\mathcal{P})).$$

Conversely, $\mathcal{A}(\mathcal{P})/\mathcal{A}_0(\mathcal{P})$ is isomorphic to $C_0(X)$, which is semisimple. This shows that the radical of $\mathcal{A}(\mathcal{P})$ is contained in $\mathcal{A}_0(\mathcal{P})$, and so

$$\mathcal{A}_0(\mathcal{P}) = J(\mathcal{A}(\mathcal{P})). \qquad \blacksquare$$

## 5. The radical of a triangular subalgebra of a continuous trace $C^*$-algebra

Suppose $X$ is a locally compact Hausdorff space, and $\langle X * \mathcal{A}, \pi\rangle$ is a bundle of elementary $C^*$-algebras over $X$, i.e. each $A(x)$ is isomorphic to the compact operators on some Hilbert space. Consider the cross sectional $C^*$-algebra of $\mathcal{A}$. Let $\mathcal{B}$ be a subbundle of Banach algebras of $\mathcal{A}$. Observe that in this case the primitive ideal space of $C_0(X, X * \mathcal{A})$ is $X$ (This follows easily from the analysis in [Di], §10.4.) The following result will be useful in the proof of our main theorem. It will be used for describing the radical of certain subalgebras of certain continuous trace $C^*$-algebras.

5.1 LEMMA: *Under the above assumptions, the Jacobson radical of $C_0(X, X*\mathcal{B})$ equals $C_0(X, X * J(\mathcal{B}))$, where $J(B(x))$ denotes the radical of $B(x)$ for each $x$ in $X$, and $J(\mathcal{B}) = \{J(B(x))\}_{x\in X}$.*

*Proof:* Observe that if $f$ is in the radical of $C_0(X, X * \mathcal{B})$, then,

$$r(fg) = 0, \quad \text{for all } g \text{ in } C_0(X, X * \mathcal{B}),$$

where $r$ denotes the spectral radius. But the equation

$$r(fg) = \lim_n \sup_{x\in X} \|(f(x)g(x))^n\|^{1/n} = 0,$$

implies

$$\lim_n \|(f(x)g(x))^n\|^{1/n} = 0, \quad \text{for all } x \text{ in } X.$$

Therefore

$$r(f(x)g(x)) = 0 \quad \text{for all } x \text{ in } X \text{ and } g(x) \text{ in } \mathcal{B}.$$

Thus $f$ is in $C_0(X, X * J(\mathcal{B}))$.

To prove the reverse inclusion observe that since $C_0(X, X * J(\mathcal{B}))$ is an ideal of $C_o(X, X * \mathcal{B})$, it suffices to show that every element $f$ in $C_0(X, X * J(\mathcal{B}))$ is quasi-nilpotent. Suppose $f \in C_0(X, X * J(\mathcal{B}))$ is not quasi-nilpotent and choose $\lambda \notin 0$ in the boundary of the spectrum $f$ as an element in $C_0(X, X * \mathcal{B})$. Then $\lambda$ lies in the spectrum of $f$ as an element of $C_0(X, X * \mathcal{A})$ ([Ri], Theorem 1.6.12). By hypothesis $\lambda$ does not belong to the spectrum of $f(x)$ with respect to $A(x)$ for every $x$ in $X$, so by definition for every $x$ in $X$, $f(x)/\lambda$ is quasi-regular, i.e. $f/\lambda$ is quasi-regular modulo each primitive ideal of $C_0(X, X * \mathcal{A})$. By Rickart ([Ri], Theorem 2.2.9), $f/\lambda$ is quasi regular in $C_0(X, X * \mathcal{A})$, so by definition, $\lambda$ is not in the spectrum of f as an element of $C_0(X, X * \mathcal{A})$. This contradiction completes the proof. ∎

5.2 THEOREM: *Let $\mathcal{E}$ be an r-discrete, principal* **T**-*groupoid over $\mathcal{G}$, assume that $C^*(\mathcal{G}; \mathcal{E})$ is a continuous trace $C^*$-algebra, and let $\mathcal{A}$ be a triangular subalgebra of $C^*(\mathcal{G}; \mathcal{E})$. Then the Jacobson radical of $\mathcal{A}$ equals the intersection of $\mathcal{A}$ with the kernel of the conditional expectation $E$ from $C^*(\mathcal{G}; \mathcal{E})$ onto $C^*(\mathcal{G})$.*

*Proof:* Suppose $C^*(\mathcal{G}; \mathcal{E})$ is continuous trace and let $X$ be its spectrum. Then $X$ is homeomorphic to $\overset{\circ}{\mathcal{G}}/\mathcal{G}$ ([MW2], Proposition 3.3). Each $x \in X$ may be viewed as $[T^u]$ for some $u$ in $\overset{\circ}{\mathcal{G}}$, where as before, $T^u$ acts on $l^2([u])$ by the formula

$$T^u(f)\xi(x) = \sum_{y \in X} K_u(x, y)\xi(y)$$

where

$$K_u(x, y) = f \circ c(x, y)\omega(x, u, y).$$

Applying [Q], we write $\mathcal{A} = \mathcal{A}(\mathcal{P})$ as the closure of the set

$$\{f \in C_c(\mathcal{G}; \mathcal{E}) | \operatorname{Supp}(f) \subseteq j^{-1}(\mathcal{P})\}$$

for $a$ suitable partial order $\mathcal{P}$. As was already observed in the proof of Theorem 3.2, if we let

$$\mathcal{P}_{[u]} := \mathcal{P} \cap [u] \times [u],$$

then

$$T^u(\mathcal{A}(\mathcal{P})) = \mathcal{A}(\mathcal{P}_{[u]}),$$

and $\mathcal{A}(\mathcal{P}_{[u]})$ is actually a subalgebra of $C^*(\mathcal{G}|_{[u]};\mathcal{E}|_{[u]})$ which is isomorphic to $C^*(\mathcal{G}|_{[u]})$. Since $\mathcal{G}|_{[u]}$ is a transitive subgroupoid of $\mathcal{G}$ the radical of $\mathcal{A}(\mathcal{P}_{[u]})$ is the set

$$\{a \in \mathcal{A}(\mathcal{P}_{[u]}) |\ a(x,x) = 0, \text{ for all } x \text{ in } [u]\}. \tag{2.1}$$

Recall that $C^*(\mathcal{G};\mathcal{E})$ is represented as $C_0(X, X * \mathcal{B})$, where $X = \mathring{\mathcal{G}}/\mathcal{G}$ is the spectrum of $C^*(\mathcal{G};\mathcal{E})$, and $\mathcal{B}$ is a bundle of elementary $C^*$-algebras over $X$. Furthermore, by (3.1)

$$\mathcal{A}(\mathcal{P}) = C_o(X, X * \mathcal{B}'),$$

where $\mathcal{B}'$ is a bundle of Banach algebras over $X$. Recall, also, that we may take

$$B'(x) = T^u(\mathcal{A}(\mathcal{P})), \qquad \text{where } [u] = x.$$

Applying 4.1, the Jacobson radical of $\mathcal{A}$ is isomorphic to $C_0(X, X * J(\mathcal{B}'))$, where

$$J(\mathcal{B}') = \{J(B'(x))\}_{x\in X}$$

and

$$J(B'(x)) = \{a \in \mathcal{A}(\mathcal{P}_{[u]}) |\ a(y,y) = 0, \text{ for all } y \text{ in } [u] = x\}.$$

Thus

$$\begin{aligned} J(\mathcal{A}(\mathcal{P})) &= \{a \in \mathcal{A}(\mathcal{P}) |\ a(y,y) = 0\} \\ &= \mathrm{Ker}(E) \cap \mathcal{A}(\mathcal{P}). \end{aligned} \qquad \blacksquare$$

## 6. The radical of a triangular subalgebra of a type I $C^*$-algebra

We now state and prove our main theorem as well as an immediate corollary.

6.1 THEOREM: *Let $\mathcal{B}$ be a type I $C^*$-algebra admitting a diagonal $\mathcal{D}$ in the sense of Kumjian, and let $\mathcal{A}$ be a triangular subalgebra of $\mathcal{B}$ with the same diagonal $\mathcal{D}$. Then the Jacobson radical of $\mathcal{A}$, $J(\mathcal{A})$, equals the intersection of $\mathcal{A}$ with the kernel of the conditional expectation $E$ from $\mathcal{B}$ onto $\mathcal{D}$. That is, $J(\mathcal{A})$ coincides with the collection of elements in $\mathcal{A}$ having "zero diagonal".*

*Proof:* Since $\mathcal{B}$ has a diagonal, there is an $r$-discrete principal $\mathbf{T}$-groupoid $\mathcal{E}$ over $\mathcal{G}$, and an isomorphism $\Phi$ from $\mathcal{B}$ onto $C^*_{\text{red}}(\mathcal{G};\mathcal{E})$ taking $\mathcal{D}$ onto $C^*(\mathring{\mathcal{G}})$ ([K], Theorem 3.1). Since $\mathcal{B}$ is type I, $\mathcal{G}$ is measurewise amenable, so $C^*{}_{\rm red}(\mathcal{G};\mathcal{E}) = C^*(\mathcal{G};\mathcal{E})$ (Remark 1.1).

On the other hand, since $\mathcal{B}$ is type I, it has a composition series $(I_\rho)_{0\le\rho\le\alpha}$ such that $I_0 = 0$, $I_\alpha = \mathcal{B}$ and $I_{\rho+1}/I_\rho$ are all continuous trace. Thus ([Re2], Corollary 4.9), there is a family of open invariant subsets of $\overset{\circ}{\mathcal{G}}$, $(U_\rho)_{0\le\rho\le\alpha}$, where $U_\rho \subseteq U_{\rho+1}$, $0 \le \rho < \alpha$, and $I_\rho = I(U_\rho)$.

Here [Re2],

$$I(U_\rho) = C^*(\mathcal{G}|_{U_\rho};\mathcal{E}|_{U_\rho}),$$

and

$$\frac{I(U_{\rho+1})}{I(U_\rho)} = C^*(\mathcal{G}|_{F_\rho};\mathcal{E}|_{F_\rho}),$$

where

$$F_{\rho+1} = U_{\rho+1}\backslash U_\rho.$$

Since $\mathcal{A}$ is a triangular subalgebra of $C^*(\mathcal{G};\mathcal{E})$, there is an open partial order $\mathcal{P}$ in $\mathcal{G}$ such that $\mathcal{A} = \mathcal{A}(\mathcal{P})$ [Q]. Let

$$\mathcal{A}_0(\mathcal{P}|_{U_\rho}) := \mathcal{A}(\mathcal{P}|_{U_\rho}) \cap \operatorname{Ker}(E).$$

Then $\mathcal{A}_0(\mathcal{P}|_{U_\rho})$ is an ideal of $\mathcal{A}(\mathcal{P}|_{U_\rho})$, and

$$\frac{\mathcal{A}_0(\mathcal{P}|_{U_{\rho+1}})}{\mathcal{A}_0(\mathcal{P}|_{U_\rho})} = \mathcal{A}_0(\mathcal{P}|_{F_{\rho+1}})$$

is a radical subalgebra of the continuous trace $C^*$-algebra $C^*(\mathcal{G}|_{F_{\rho+1}};\mathcal{E}|_{F_{\rho+1}})$ (see 5.3).

Observe that $I(U_1)$ is a continuous trace $C^*$-algebra and hence (see 5.2):

$$J(\mathcal{A}(\mathcal{P}|_{U_1})) = \mathcal{A}_0(\mathcal{P}|_{U_1}).$$

We use transfinite induction. Assume now that $\rho$ is not a limit ordinal, and that

$$(*) \qquad J(\mathcal{A}(\mathcal{P}|_{U_{\rho'}})) = \mathcal{A}_0(\mathcal{P}|_{U_{\rho'}}), \qquad \text{for all } \rho' < \rho.$$

We have

$$J(\mathcal{A}_0(\mathcal{P}|_{U_\rho})) = J(\mathcal{A}(\mathcal{P}|_{U_\rho})) \subseteq \mathcal{A}_0(\mathcal{P}|_{U_\rho}),$$

due to the semisimplicity of $\frac{\mathcal{A}(\mathcal{P}|_{U_\rho})}{\mathcal{A}_0(\mathcal{P}|_{U_\rho})}$. (Remember this quotient is isomorphic to $C^*(U_\rho) = C_0(U_\rho)$.) On the other hand

$$\mathcal{A}_0(\mathcal{P}|_{U_{\rho-1}}) \subseteq J(\mathcal{A}(\mathcal{P}|_{U_\rho})),$$

since $\mathcal{A}_0(\mathcal{P}|_{U_{\rho-1}})$ is an ideal of $\mathcal{A}(\mathcal{P}|_{U_\rho})$ consisting of quasi-nilpotent elements. Hence

$$\mathcal{A}_0(\mathcal{P}|_{U_{\rho-1}}) \subseteq J(\mathcal{A}_0(\mathcal{P}|_{U_\rho}).$$

Therefore

$$\frac{\mathcal{A}_0(\mathcal{P}|_{U_\rho})}{J(\mathcal{A}_0(\mathcal{P}|_{U_\rho}))} \text{ is the homomorphic image of } \frac{\mathcal{A}_0(\mathcal{P}|_{U_\rho})}{\mathcal{A}_0(\mathcal{P}|_{U_{\rho-1}})}$$

by the homomorphism that sends each coset $a + \mathcal{A}_0(\mathcal{P}|_{U_{\rho-1}})$ to $a + J(\mathcal{A}_0(\mathcal{P}|_{U_\rho}))$. But since the latter is a radical algebra, and the first one is semisimple, we conclude that

$$\frac{\mathcal{A}_0(\mathcal{P}|_{U_\rho})}{J(\mathcal{A}_0(\mathcal{P}|_{U_\rho}))} = 0,$$

and so

$$J(\mathcal{A}(\mathcal{P}|_{U_\rho})) = \mathcal{A}_0(\mathcal{P}|_{U_\rho}).$$

Suppose now that $\rho$ is a limit ordinal and that $(*)$ holds. We show that the same argument works also in this case. Here $I_\rho$ is defined to be the closure of $\bigcup_{\rho'<\rho} I_{\rho'}$. Then

$$J(\mathcal{A}(\mathcal{P}|_{U_\rho})) \supseteq \mathcal{A}_0(\mathcal{P}) \cap \bigcup_{\rho'<\rho} I_{\rho'}$$

and since $J(\mathcal{A}_0(\mathcal{P}|_{U_\rho}))$ is a closed ideal of $\mathcal{A}(\mathcal{P}|_{U_\rho})$, it follows that

$$J(\mathcal{A}_0(\mathcal{P}|_{U_\rho})) \supseteq \mathcal{A}_0(\mathcal{P}|_{U_{\rho'}}), \quad \rho' < \rho.$$

Also

$$J(\mathcal{A}_0(\mathcal{P}|_{U_\rho})) = J(\mathcal{A}(\mathcal{P}|_{U_\rho})) \subseteq \mathcal{A}_0(\mathcal{P}|_{U_\rho}),$$

due to the semisimplicity of $\frac{\mathcal{A}(\mathcal{P}|_{U_\rho})}{\mathcal{A}_0(\mathcal{P}|_{U_\rho})}$. Therefore

$$\frac{\mathcal{A}_0(\mathcal{P}|_{U_\rho})}{J(\mathcal{A}_0(\mathcal{P}|_{U_\rho}))} \text{ is the homomorphic image of } \frac{\mathcal{A}_0(\mathcal{P}|_{U_\rho})}{\mathcal{A}_0(\mathcal{P}|_{U_{\rho'}})},$$

and we conclude that

$$J(\mathcal{A}(\mathcal{P}|_{U_\rho})) = \mathcal{A}_0(\mathcal{P}|_{U_\rho}). \qquad \blacksquare$$

We state now a corollary which follows immediately from the preceding theorem. In the case in which the $C^*$-algebra $\mathcal{B}$ is the analytic crossed product, this corollary generalizes Corollary 4.2.1 [M1]. We will see this in more detail in the next section.

6.2 COROLLARY: *Let $\mathcal{B}$ be a non-antiliminar $C^*$-algebra admitting a diagonal $\mathcal{D}$, assume the associated groupoid is measurewise amenable, and let $\mathcal{A}$ be a triangular subalgebra of $\mathcal{B}$ with the same diagonal $\mathcal{D}$. Then the radical of $\mathcal{A}$ is non-zero.*

*Proof:* Assume $\mathcal{B}$ is isomorphic to $C^*(\mathcal{G};\mathcal{E})$ and let $I$ denote the type I part of $\mathcal{B}$. Then $I$ is non-zero, and by [Re2] there exists an open invariant subset of $\mathring{\mathcal{G}}$ such that $I = I(U)$. Therefore, if $\mathcal{A} = \mathcal{A}(\mathcal{P})$, then $\mathcal{A} \cap I$ is a nonzero ideal of $\mathcal{A}$ whose Jacobson radical equals $\mathcal{A}_0 \cap I$. Set $I_0 = \mathcal{A}_0 \cap I$. Then

$$J(I_0) = I_0 \cap J(\mathcal{A}) = I_0. \quad \blacksquare$$

*6.3 Remark:* We do not know if 6.2 is valid without the assumption that the groupoid is measurewise amenable. We know that in general the correspondence between ideals in $C^*(\mathcal{G};\mathcal{E})$ and open invariant subsets of $\mathring{\mathcal{G}}$ breaks down in the nonamenable case, but we do not know whether if $C^*(\mathcal{G};\mathcal{E})$ is not antiliminary then there is an open invariant set $U \subseteq \mathring{\mathcal{G}}$ such that $C^*(\mathcal{G}|_U;\mathcal{E}|_U)$ is type I. This is really all we need. $\blacksquare$

## 7. Analytic crossed products

In this section we give some examples and applications of the previous results. Let $X$ be a locally compact Hausdorff space, and let $\tau$ be a homeomorphism of $X$. Then a $\mathbf{Z}$ action on $X$ is defined, where $(n,x) \to \tau^n x$. Since every homeomorphism $\tau$ of $X$ determines an automorphism of $C_0(X)$, which we are going also to call $\tau$, a $C^*$-dynamical system $(C_0(X),\mathbf{Z},\tau)$ is defined. This dynamical system determines a $C^*$-algebra called the **$C^*$-algebra crossed product** and is also denoted $C^*(X,\mathbf{Z})$ or $C^*(X,\tau)$. Suppose that the action is free, i.e. that $\tau^n x \notin x$ for all $x$ in $X$ and all $n \neq 0$. Observe that $C^*(X,Z)$ admits a diagonal which is isomorphic to $C_0(X)$. Consider the subalgebra $\mathcal{A}(X,\tau)$ which is the closure in the $C^*$-norm of those functions in $C_c(X \times \mathbf{Z})$ supported in $X \times \mathbf{Z}_+$. Then $\mathcal{A}(X,\mathbf{Z})$ is a triangular subalgebra of $C^*(X,\mathbf{Z})$ with respect to $C_0(X)$ called the **analytic crossed product** determined by the dynamical system. Muhly and Peters in [M1], [P1] and [P2] addressed the problem of finding the radical of the analytic crossed product in this particular case. We have just given a partial answer to that question in a more general context. That is, Theorem 6.1 identifies the radical of $\mathcal{A}(X,\mathbf{Z})$ when $C^*(X,\mathbf{Z})$ is type I. More generally, let $H$ be any discrete group acting freely on a locally compact Hausdorff space $X$ such that $C^*(X,H)$ is type I, and let $\Sigma$ be a subsemigroup of $H$ such that $\Sigma \cap \Sigma^{-1} = \{e\}$, where $e$ is the identity in $H$. If we denote by $\mathcal{A}(\Sigma)$ the analytic crossed product, then the radical of $\mathcal{A}(\Sigma)$ is described in Theorem 6.1. (For a definition of the analytic crossed product we refer the reader to [MM].) In this case the groupoid is the orbit equivalence relation $\{(x, tx) | x \in X, t \in \Sigma\}$. The algebra $\mathcal{A}(\Sigma)$ is $\mathcal{A}(\mathcal{P})$ and we see from Theorem 6.1 that

$$J(\mathcal{A}(\Sigma)) = J(\mathcal{A}(\mathcal{P})) = \{f \in \mathcal{A}(\mathcal{P}) |\ f(x,e) = 0\}.$$

(Remember, elements in $C^*(X,H)$ may be viewed as functions on $X \times H$, under the hypothesis that $C^*(X,H)$ is type I, and a function $f$ lies in $\mathcal{A}(\Sigma)$ if and only if $f$ is supported in $X \times \Sigma$ .)

To get a little clearer feel for our analysis recall that if $H$ is a group acting on $X$, a compact subset $K$ of $X$ is called **wandering** iff the set $\{t \,|\, tK \cap K \neq \emptyset\}$ is compact in $H$. A point $x$ in $X$ is called **wandering** if it has a wandering neighborhood. Observe that the set of wandering points is an open and invariant subset of $X$.

Suppose we have a dynamical system $(X,H)$. Let $X^*$ denote the subset of wandering points of $X$. Call it $W_1$. Take now the complement of $X^*$ in $X$. Call it $X_1$ and let $X_1^*$ be the set of wandering points of $X_1$. Let $W_2 = X^* \cup X_1^*$. If we proceed this way using transfinite induction, we get a chain of open invariant subspaces of $X$: $0 = W_0 \subsetneq W_1 \subsetneq W_2 \subsetneq \cdots$. When the space $X$ is locally compact and Hausdorff the chain stops in countably many steps ([GH] 7.19 and 7.20). Suppose $W_\gamma$ is the open subspace of $X$ such that $W_\alpha = W_\gamma$ for all $\alpha > \gamma$. Then the complement in $X$ of $W_\gamma$ is called the **center** of the transformation group.

A result by Green [Gr] shows that $C^*(X,H)$ has continuous trace if and only if compact subsets of $X$ are wandering. This result was later generalized by Muhly and Williams ([MW1] and [MW2]). Based on this result, observe the following: (1) The sets $(W_\rho)_{0\leq\rho\leq\alpha}$ are open, and therefore ([Re 3] Corollary 4.9) they correspond to an increasing family of ideals $(I_\rho)_{0\leq\rho\leq\alpha}$, where $I_\rho = I(W_\rho)$ in $C^*(X,H)$. (2) Each quotient is a continuous trace $C^*$-algebra (since $I(W_\rho)/I(W_{\rho-1})$ is isomorphic to $I((X\backslash W_{\rho-1})^*)$. (3) $C^*(X,H)/I_\gamma$ is antiliminary. This means that $C^*(W_\gamma, H)$ corresponds to the type I part of $C^*(X,H)$. If $W_\gamma$ is not empty (and if $H$ is discrete), then the radical of $\mathcal{A}(\Sigma)$ is non-zero (6.3). If $C^*(X,H)$ is type

I, then the chain of ideals described above is the canonical composition series for $C^*(X, H)$.

To illustrate the analysis just given, we have chosen an example of a dynamical system whose corresponding $C^*$-algebra is type I but is not continuous trace. The example of the dynamical system was taken from [BS].

*7.1 Example:* Consider a dynamical system in an euclidean $(x_1, x_2)$-plane, whose phase portrait is as in Fig.1. Although the action here is that of a continuous group, we can think of each trajectory as being discrete.

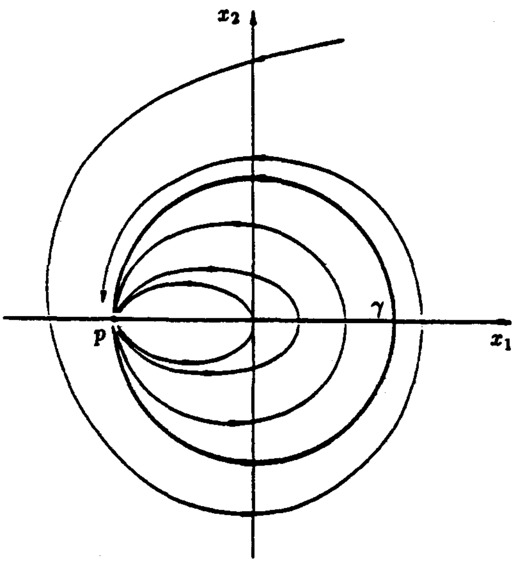


Fig. 1.

The unit circle contains a rest point $p$ and an orbit $\gamma$ such that no point $q$ in $\gamma$ has a wandering neighborhood. Consider the dynamical system obtained from this one by deleting the rest point $p$ (the dynamical system is thus defined on $\mathbf{R}^2 \backslash \{p\}$). It is easily seen that the transformation group $C^*$-algebra corresponding to $(\mathbf{R}^2 \backslash \{p\}, \mathbf{Z})$ is type I but not continuous trace. In fact the ideal of continuous trace elements of this $C^*$-algebra is the closure in the $C^*$-norm of those functions $f$ in $C_c(\mathbf{R}^2, \mathbf{Z})$ such that supp($f$) is contained in $X \backslash \gamma \times \mathbf{Z}$, where $X = \mathbf{R}^2 \backslash \{p\}$. By 6.2, the radical of $\mathcal{A}(\mathbf{Z}_+)$ is the closure of the set of funcions supported in $X \times \mathbf{Z}_+ \backslash \{0\}$.

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## References

[A] W. B. Arveson, *Operator algebras and invariant subspaces*, Ann. of Math. **100** (1974), 433–532.

[AJ] W. B. Arveson and K. B. Josephson, *Operator algebras and measure preserving automorphisms II*, J. Functional Analysis **4** (1969), 100–134.

[BD] F. F. Bonsall and J. Duncan, *Complete Normed Algebras*, Springer-Verlag, New York, Heidelberg, Berlin, 1973.

[BS] N. P. Bhatia and G. P. Szegö, *Stability Theory of Dynamical Systems*, Springer-Verlag, New York, Heidelberg, Berlin, 1970.

[Da] K. J. Davidson, *Nest Algebras*, Pure Mathematics Department, University of Waterloo, Waterloo, Ontario, Canada.

[Di] J. Dixmier, *Les $C^*$-algèbres et leurs représentations*, Gauthier-Villars, Paris, 1964.

[FD] J. M. G. Fell and R. S. Doran, *Representation of *-Algebras, Locally Compact Groups, and Banach *-Algebraic Bundles*, Volume I, Academic Press, New York, 1988.

[GH] W. H. Gottschalk and G. A. Hedlund, *Topological Dynamics*, American Mathematical Society, Colloquium Publications, Volume XXVI, 1955.

[Go] R. Godement, *Sur la théorie des répresentations unitaires*, Ann. Math. **53** (1951), 68–124.

[Gr] P. Green, *$C^*$-algebras of smooth transformation groups with smooth orbit space*, Pac. J. Math. **72** (1977), 71–96.

[K] A. Kumjian, *On $C^*$-diagonals*, Canad. J. Math. **38** (1969), 969–1008.

[M1] P. S. Muhly, *Radicals, crossed products, and flows*, Ann. Polon. Math. **43** (1983), 35–42.

[M2] P. S. Muhly, *CBSM Regional Conference Series*, preprint.

[MM] M. J. McAsey and P. S. Muhly, *Representations of non-self-adjoint crossed products*, Proc. London Math. Soc.(3) **47** (1983), 128–144.

[MS] P. S. Muhly and B. Solel, *Subalgebras of groupoid $C^*$-algebras*, J. Reine Angew. Math. **402** (1989), 41–75.

[MSS] P. S. Muhly, K. Saito and B. Solel, *Coordinates for triangular operator algebras*, Ann. Math. **127** (1988), 245–278.

[MW1] P. S. Muhly and D. P. Williams, *Continuous trace groupoid $C^*$-algebras*, Math. Scand. **66** (1990), 231–241.

[MW2] P. S. Muhly and D. P. Williams, *Continuous trace groupoid $C^*$-algebras II*, Math. Scand., preprint.

[P1] J. Peters, *Semi-crossed products of $C^*$-algebras*, J. Functional Analysis **59** (1984), 498–534.

[P2] J. Peters, *The ideal structure of certain nonselfadjoint operator algebras*, Trans. Amer. Math. Soc. **305** (1988), 333–352.

[Q] C. Qiu, *Ph.D. Dissertation*, The University of Iowa, August 1991.

[Ra] A. Ramsay, *The Mackey–Glimm dichotomy for foliations and other Polish groupoids*, J. Functional Analysis **94** (1990), 358–374.

[Re1] J. Renault, *A groupoid approach to $C^*$-algebras*, Lecture Notes in Mathematics, No. 793, Springer-Verlag, New York, 1980.

[Re2] J. Renault, *Rerésentations des produits croisés d'algèbres de groupoïdes*, J. Operator Theory **18** (1987), 67–97.

[Re3] J. Renault, *Structure des ideaux des produits croisés de $C^*$-algèbres de groupoïdes*, Publications Mathématiques de l'Université Pierre et Marie Curie, 1988.

[Ri] C. Rickart, *General Theory of Banach Algebras*, D. Van Nostrand Company, Inc, Princeton, 1960.